# Supersymmetric field theories

**Antoine Van Proeyen**

**KU Leuven**

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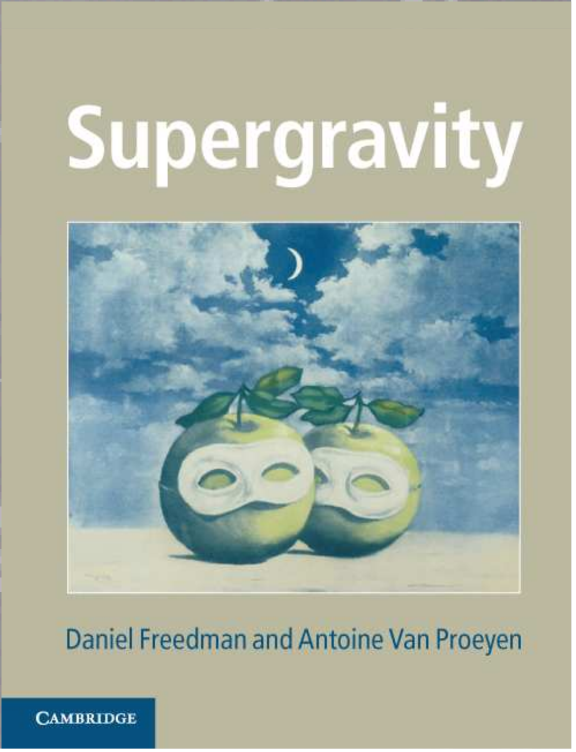
Based on some chapters of the book
'Supergravity'



1973

Wess, Zumino

# Supersymmetry and supergravity

- supersymmetry

  Bosons and fermions
  in one multiplet

$$\delta A(x) = \bar{\varepsilon}\,\psi(x)$$

$$\delta \psi(x) = \gamma^{\mu} \varepsilon \frac{\partial}{\partial x^{\mu}} A(x)$$

- commutator gives general coordinate transformations

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)] = \overline{\varepsilon_2}\, \gamma^{\mu} \varepsilon_1 \frac{\partial}{\partial x^{\mu}} \qquad \text{or} \qquad \{Q, Q\} = \gamma^{\mu} P_{\mu}$$

$\Rightarrow$ gauge theory contains gravity: Supergravity

1976

Freedman, van Nieuwenhuizen, Ferrara

# 1. Scalar field theory and its symmetries:

## A. Poincaré group

Space with $(x^\mu) = (t, \vec{x})$

Metric

$$\mathrm{d}s^2 = -\mathrm{d}t\mathrm{d}t + \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x} = \mathrm{d}x^\mu \eta_{\mu\nu}\mathrm{d}x^\nu$$

Isometries (preserve metric)

$$x^\mu = \Lambda^\mu{}_\nu x'^\nu + a^\mu$$

$$\Lambda^\mu{}_\rho \eta_{\mu\nu}\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$$

Expand

$$\begin{aligned} \Lambda^\mu{}_\nu &= \delta^\mu{}_\nu + \lambda^\mu{}_\nu + \mathcal{O}(\lambda^2) \\ &= \left(\mathrm{e}^{\frac{1}{2}\lambda^{\rho\sigma}m_{[\rho\sigma]}}\right)^\mu{}_\nu \end{aligned}$$

$$m_{[\rho\sigma]}{}^\mu{}_\nu \equiv \delta^\mu_\rho \eta_{\nu\sigma} - \delta^\mu_\sigma \eta_{\rho\nu} = -m_{[\sigma\rho]}{}^\mu{}_\nu$$

Algebra SO(1, D-1)

$$\begin{aligned} [m_{[\mu\nu]}, m_{[\rho\sigma]}] &= \eta_{\nu\rho}m_{[\mu\sigma]} - \eta_{\mu\rho}m_{[\nu\sigma]} \\ &\quad -\eta_{\nu\sigma}m_{[\mu\rho]} + \eta_{\mu\sigma}m_{[\nu\rho]} \end{aligned}$$

Act on fields: $\phi$(x)=$\phi'$(x')

$$\begin{aligned} \phi'(x) &= U(\Lambda)\phi(x) = \phi(\Lambda x) \\ U(\Lambda) &\equiv \mathrm{e}^{-\frac{1}{2}\lambda^{\rho\sigma}L_{[\rho\sigma]}} \\ L_{[\rho\sigma]} &\equiv x_\rho\partial_\sigma - x_\sigma\partial_\rho \end{aligned}$$

More general if not scalar fields

$$J_{[\rho\sigma]} = L_{[\rho\sigma]}\mathbb{1} + m_{[\rho\sigma]}\,,$$

$$\begin{aligned} \psi'^i(x) &= U(\Lambda, a)^i{}_j\psi^j(x) \\ &= \left(\mathrm{e}^{-\frac{1}{2}\lambda^{\rho\sigma}m_{[\rho\sigma]}}\right)^i{}_j\psi^j(\Lambda x + a) \end{aligned}$$

# B. Other symmetries and currents

Generic infinitesimal

$$\delta\phi^i(x) \equiv \epsilon^A \Delta_A \phi^i(x)\,,$$

(constant parameters).

Transformation of Lagrangian:

$$\delta\mathcal{L} \equiv \epsilon^A \left[ \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\partial_\mu\Delta_A\phi^i + \frac{\delta\mathcal{L}}{\delta\phi^i}\Delta_A\phi^i \right] = \epsilon^A \partial_\mu K^\mu_A\,.$$

Leads to conserved currents

$$J^\mu{}_A = -\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\Delta_A\phi^i + K^\mu_A\,, \qquad \partial_\mu J^\mu{}_A \approx 0\,.$$

# Exercises on chapter 1

■ Ex 1.5: Show that the action

$$S=\int \mathrm{d}^D x\,\mathcal{L}(x)=-\tfrac{1}{2}\int \mathrm{d}^D x\left[\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^i+m^2\phi^i\phi^i\right]$$

is invariant under the transformation

$$\phi^i(x)\xrightarrow{\Lambda}\phi'^i(x)\equiv\phi^i(\Lambda x).$$

Important: fields transform, not the integration variables

■ Ex.1.6: Compute the commutators $[L_{[\mu\nu]},L_{[\rho\sigma]}]$ and show that they agree with that for matrix generators. Show that to first order in $\lambda^{\rho\sigma}$

$$\phi^i(x^\mu)-\tfrac{1}{2}\lambda^{\rho\sigma}L_{[\rho\sigma]}\phi^i(x^\mu)=\phi^i(x^\mu+\lambda^{\mu\nu}x_\nu)$$

# 2. The Dirac field

$$\partial\!\!\!/\Psi(x) \equiv \gamma^\mu \partial_\mu \Psi(x) = m\Psi(x)\,.$$

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\,\eta^{\mu\nu}\,\mathbb{1}$$

Lorentz transformations generated by

$$\Sigma^{\mu\nu} \equiv \tfrac{1}{4}\,[\gamma^\mu, \gamma^\nu]\,,$$

which satisfies Lorentz algebra.

For actions we need

$$\bar\Psi = \Psi^\dagger \beta = \Psi^\dagger \mathrm{i}\gamma^0\,,$$

such that spinor bilinears can be formed that are Lorentz invariants:

$$\delta\Psi = -\tfrac{1}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu}\Psi\,, \qquad \delta\bar\Psi = \tfrac{1}{2}\lambda^{\mu\nu}\bar\Psi\Sigma_{\mu\nu}$$

# Exercise on chapter 2

- Show using the fundamental relation of gamma matrices that

$$[\Sigma^{\mu\nu},\gamma^\rho] = 2\gamma^{[\mu}\eta^{\nu]\rho} = \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\mu\rho}$$

- Prove the consistency of

$$\delta\Psi = -\tfrac{1}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu}\Psi\,, \qquad \delta\bar{\Psi} = \tfrac{1}{2}\lambda^{\mu\nu}\bar{\Psi}\Sigma_{\mu\nu}$$

- Prove then the invariance of the action

$$S[\bar{\Psi},\Psi] = -\int \mathrm{d}^D x\,\bar{\Psi}[\gamma^\mu\partial_\mu - m]\Psi(x)$$

# 3. Clifford algebras and spinors

- Determines the properties of
  - the spinors in the theory
  - the supersymmetry algebra
- We should know
  - how large are the smallest spinors in each dimension
  - what are the reality conditions
  - which bispinors are (anti)symmetric (can occur in superalgebra)

# 3.1 The Clifford algebra in general dimension

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\,\eta^{\mu\nu}\,\mathbb{1}$$

## 3.1.1 The generating $\gamma$ matrices

Hermiticity $\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$ (hermitian for spacelike)

representations related by conjugacy by unitary $S$

$$\gamma'^\mu = S \gamma^\mu S^{-1}$$

## 3.1.2 The complete Clifford algebra

$$\gamma^{\mu_1\ldots\mu_r} = \gamma^{[\mu_1}\ldots\gamma^{\mu_r]}\,, \qquad \text{e.g.} \qquad \gamma^{\mu\nu} = \tfrac{1}{2}\gamma^\mu\gamma^\nu - \tfrac{1}{2}\gamma^\nu\gamma^\mu$$

## 3.1.3 Levi-Civita symbol

$$\varepsilon_{012(D-1)} = 1\,, \qquad \varepsilon^{012(D-1)} = -1$$

## 3.1.4 Practical $\gamma$ -gamma matrix manipulation

$$\gamma^\mu\gamma_\mu = D\,, \qquad \gamma^{\mu\nu}\gamma_\nu = (D-1)\gamma^\mu$$

# 3.1.5 Basis of the algebra for even dimension $D = 2\,m$

$$\{\Gamma^A = \mathbb{1}, \gamma^\mu, \gamma^{\mu_1\mu_2}, \gamma^{\mu_1\mu_2\mu_3}, \ldots, \gamma^{\mu_1\cdots\mu_D}\}$$

with $\mu_1 < \mu_2 < \ldots < \mu_r$

reverse order list

$$\{\Gamma_A = \mathbb{1}, \gamma_\mu, \gamma_{\mu_2\mu_1}, \gamma_{\mu_3\mu_2\mu_1}, \ldots, \gamma_{\mu_D\cdots\mu_1}\}.$$

$$\mathrm{Tr}(\Gamma^A\Gamma_B) = 2^m\,\delta_B^A$$

expansion for any matrix in spinor space $M$

$$M = \sum_A m_A\Gamma^A, \qquad m_A = \frac{1}{2^m}\mathrm{Tr}(M\Gamma_A)$$

# 3.1.6 The highest rank Clifford algebra element

$$\gamma_* \equiv (-\mathrm{i})^{m+1}\gamma_0\gamma_1\cdots\gamma_{D-1}\,,$$

which satisfies $\gamma_*^2 = \mathbb{1}$.
E.g. $D = 4$: $\gamma_* = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$.

Projections

$$P_L = \tfrac{1}{2}(\mathbb{1} + \gamma_*)\,, \qquad P_R = \tfrac{1}{2}(\mathbb{1} - \gamma_*)\,.$$

# 3.1.7 Odd spacetime dimension *D=2m+1*

$\gamma$ matrices dan be constructed in two ways from those in *D=2m*:

$$\gamma_{\pm}{}^{\mu} = (\gamma^{0}, \gamma^{1}, \ldots, \gamma^{(2m-1)}, \gamma^{2m} = \pm\gamma_{*})$$

The set with all $\gamma^{\mu_1\cdots\mu_r}$ is overcomplete

$$\gamma_{\pm}^{\mu_1\cdots\mu_r} = \pm \mathrm{i}^{m+1}\frac{1}{(D-r)!}\varepsilon^{\mu_1\cdots\mu_D}\gamma_{\pm\,\mu_D\cdots\mu_{r+1}}$$

# 3.2 Supersymmetry and symmetry of bi-spinors (intro)

- E.g. a supersymmetry on a scalar is a symmetry transformation depending on a spinor ε:

$$\delta(\epsilon)\phi(x) = \bar{\epsilon}\,\psi(x)$$

- For the algebra we should obtain a GCT

$$[\delta(\epsilon_2), \delta(\epsilon_1)]\,\phi(x) = \underbrace{\bar{\epsilon}_1\gamma^\mu\epsilon_2}_{\xi^\mu}\partial_\mu\phi(x)$$

- Then the GCT parameter should be antisymmetric in the spinor parameters

$$\xi^\mu = \bar{\epsilon}_1\gamma^\mu\epsilon_2 = -\bar{\epsilon}_2\gamma^\mu\epsilon_1$$

Thus, to see what is possible, we have to know the symmetry properties of bi-spinors

# 3.2 Spinors in general dimensions

## 3.2.1 Spinors and spinor bilinears

Majorana conjugate

$$\bar{\lambda} = \lambda^T C$$

$C$ is a matrix such that $C\gamma_{\mu_1 \dots \mu_r}$ are all symmetric or antisymmetric, depending only on $D$ and $r$.

- with anticommuting spinors

$$\bar{\lambda}\gamma_{\mu_1 \dots \mu_r}\chi = t_r \bar{\chi}\gamma_{\mu_1 \dots \mu_r}\lambda$$

| D (mod 8) | $t_r = -1$ | $t_r = +1$ |
|---|---|---|
| 0 | 0,3 | 2,1 |
| | **0,1** | **2,3** |
| | 0,1 | 2,3 |
| | 0,1 | 2,3 |
| | **1,2** | **0,3** |
| | 1,2 | 0,3 |
| 4 | **1,2** | **0,3** |
| | 2,3 | 0,1 |
| 5 | **2,3** | **0,1** |
| 6 | 2,3 | 0,1 |
| | **0,3** | **1,2** |
| 7 | 0,3 | 1,2 |

Since symmetries of spinor bilinears are important for supersymmetry, we use the Majorana conjugate to define $\bar{\lambda}$.

# 3.2.2 Spinor indices

$$\lambda^\alpha = \mathcal{C}^{\alpha\beta}\lambda_\beta\,, \qquad \lambda_\alpha = \lambda^\beta\mathcal{C}_{\beta\alpha}\,.$$

NW-SE convention

Note that $\mathcal{C}_{\alpha\beta}$ are components of $C^{-1}$ and $\mathcal{C}^{\alpha\beta}$ of $C^T$.

Translations:

$$\bar{\chi}\gamma_\mu\lambda = \chi^\alpha(\gamma_\mu)_\alpha{}^\beta\lambda_\beta\,,$$

and also

$$(\gamma_\mu)_{\alpha\beta} = (\gamma_\mu)_\alpha{}^\gamma\mathcal{C}_{\gamma\beta}$$

Have symmetry $-t_1$: $(\gamma_\mu)_{\alpha\beta} = -t_1(\gamma_\mu)_{\beta\alpha}$.

# 3.2.4 Reality

Complex conjugation can be replaced by charge conjugation, an operation that acts as complex conjugation on scalars, and has a simple action on fermion bilinears. For example, it preserves the order of spinor factors.

In fact complex conjugation uses

$$\gamma^{\mu *} = -t_0 t_1 \, B\gamma^\mu B^{-1} \,, \qquad B \equiv \mathrm{i} t_0 C\gamma^0$$

We use

$$\lambda^C \equiv B^{-1}\lambda^* \,, \qquad (\gamma_\mu)^C \equiv B^{-1}\gamma_\mu^* B = (-t_0 t_1)\gamma_\mu \,.$$

It works like this:

$$(\bar\chi M\lambda)^* \equiv (\bar\chi M\lambda)^C = (-t_0 t_1)\overline{\chi^C} M^C \lambda^C$$

# 3.3 Majorana spinors

- A priori a spinor $\psi$ has $2^{\mathrm{Int}[D/2]}$ (complex) components
- Using e.g. 'left' projection $P_L = (1+\gamma_*)/2$

  'Weyl spinors' $P_L\ \psi = \psi$ **if *D* is even** (otherwise trivial)
- In some dimensions (and signature) there are reality conditions

  $\psi = \psi^C = B^{-1}\ \psi^*$

  consistent with Lorentz algebra: 'Majorana spinors'
- consistency requires $t_1 = -1$.

# Other types of spinors

- If $t_1$=1: Majorana condition not consistent
- Define other reality condition (for an even number of spinors):

$$\chi^i = \varepsilon^{ij}(\chi^j)^C$$

- 'Symplectic Majorana spinors'
- In some dimensions Weyl and Majorana can be combined, e.g. reality condition for Weyl spinors: 'Majorana-Weyl spinors'

$D = 2$ mod 8:

Majorana: $\psi^C = \psi\,,$ Weyl: $P_{L,R}\psi = \psi$

$D = 4$ mod 4

$$(P_L\psi)^C = P_R\psi\,, \qquad (P_R\psi)^C = P_L\psi$$

Possibilities for susy depend on the properties of irreducible spinors in each dimension

- Dependent on signature. Here: Minkowski
- M: Majorana
  MW: Majorana-Weyl
  S: Symplectic
  SW: Symplectic-Weyl

| Dim | Spinor | min.# comp |
|---|---|---|
| 2 | MW | 1 |
| 3 | M | 2 |
| 4 | M | 4 |
| 5 | S | 8 |
| 6 | SW | 8 |
| 7 | S | 16 |
| 8 | M | 16 |
| 9 | M | 16 |
| 10 | MW | 16 |
| 11 | M | 32 |

# 3.4 Majorana OR Weyl fields in D=4

- Any field theory of a Majorana spinor field $\Psi$ can be rewritten in terms of a Weyl field $P_L\Psi$ and its complex conjugate.
- Conversely, any theory involving the chiral field $\chi=P_L\chi$ and its conjugate $\chi^C=P_R\chi^C$ can be rephrased as a Majorana equation if one defines the Majorana field $\Psi=P_L\chi+P_R\chi^C$.
- Supersymmetry theories in D=4 are formulated in both descriptions in the physics literature.

# Exercise on chapter 3

- Ex. 3.40: Rewrite

$$S[\Psi] = -\frac{1}{2}\int \mathrm{d}^D x\, \bar{\Psi}[\gamma^\mu \partial_\mu - m]\Psi(x)$$

as

$$\begin{aligned} S[\psi] &= -\tfrac{1}{2}\int \mathrm{d}^4 x\, [\bar{\Psi}\gamma^\mu\partial_\mu - m]\,(P_L + P_R)\Psi \\ &= -\int \mathrm{d}^4 x\, \left[\bar{\Psi}\gamma^\mu\partial_\mu P_L\Psi - \tfrac{1}{2}m\bar{\Psi}P_L\Psi - \tfrac{1}{2}m\bar{\Psi}P_R\Psi\right] . \end{aligned}$$

and prove that the Euler-Lagrange equations are

$$\not{\partial} P_L\Psi = mP_R\Psi\,, \qquad \not{\partial} P_R\Psi = mP_L\Psi\,.$$

Derive $\Box\, P_{L,R}\Psi = m^2\, P_{L,R}\Psi$ from the equations above

# 4. The Maxwell and Yang-Mills Gauge Fields

## 4.1 The Abelian gauge field $A_\mu(x)$

Couple to

$$\Psi(x) \to \Psi'(x) \equiv \mathrm{e}^{\mathrm{i}q\theta(x)}\Psi(x)\,.$$

Due to

$$A_\mu(x) \to A'_\mu(x) \equiv A_\mu(x) + \partial_\mu\theta(x)\,.$$

with covariant derivatives

$$D_\mu\Psi(x) \equiv (\partial_\mu - \mathrm{i}qA_\mu(x))\Psi(x)\,,$$

Field strengths couple to currents

$$\partial^\mu F_{\mu\nu} = -J_\nu\,, \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

# 4.3 Non-abelian gauge symmetry

- Simplest: act by matrices and $[t_A, t_B] = f_{AB}{}^C t_C$
- Gauge fields for any generator

  cov. derivative: $D_\mu \Psi = \left(\partial_\mu + g t_A A_\mu^A\right)\Psi$

  needs transform: $\delta A_\mu^A(x) = \frac{1}{g}\partial_\mu \theta^A + \theta^C(x) A_\mu^B(x) f_{BC}{}^A$

- Curvatures $[D_\mu, D_\nu]\Psi = g F_{\mu\nu}^A t_A \Psi$

$$F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + g f_{BC}{}^A A_\mu^B A_\nu^C$$

- Typical action

$$S[A_\mu^A, \bar\Psi_\alpha, \Psi^\alpha] = \int \mathrm{d}^D x \left[-\tfrac{1}{4} F^{A\mu\nu} F_{\mu\nu}^A - \bar\Psi_\alpha(\gamma^\mu D_\mu - m)\Psi^\alpha\right]$$

# Exercise on chapter 4

- Ex. 4.17: Use the Jacobi identity to show that the matrices $(t_A)^D{}_E = f_{AE}{}^D$ satisfy $[t_A, t_B] = f_{AB}{}^C t_C$ and therefore give a representation
- Ex 4.21: Show that

$$D_\mu F^A_{\nu\rho} + D_\nu F^A_{\rho\mu} + D_\rho F^A_{\mu\nu} = 0$$

is satisfied identically if $F_{\mu\nu}{}^A$ is written in the form

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + g f_{BC}{}^A A^B_\mu A^C_\nu$$

# 6. N=1 Global supersymmetry in D=4

- Classical algebra $\{Q_\alpha, Q_\beta\} = -\frac{1}{2}\gamma^{\mu}_{\alpha\beta}P_\mu$

$$[P, Q] = 0$$

$$[M_{\mu\nu}, Q] = -\frac{1}{2}\gamma_{\mu\nu}Q$$

# 6.2. SUSY field theories of the chiral multiplet

- Transformation under SUSY

$$\begin{aligned} \delta Z &= \frac{1}{\sqrt{2}}\bar\epsilon P_L\chi\,, \\ \delta P_L\chi &= \frac{1}{\sqrt{2}}P_L(\partial\!\!\!/ Z + F)\epsilon\,, \\ \delta F &= \frac{1}{\sqrt{2}}\bar\epsilon\,\partial\!\!\!/ P_L\chi \end{aligned}$$

$$\begin{aligned} \delta \bar Z &= \frac{1}{\sqrt{2}}\bar\epsilon P_R\chi\,, \\ \delta P_R\chi &= \frac{1}{\sqrt{2}}P_R(\partial\!\!\!/ \bar Z + \bar F)\epsilon\,, \\ \delta \bar F &= \frac{1}{\sqrt{2}}\bar\epsilon\,\partial\!\!\!/ P_R\chi \end{aligned}$$

- Algebra

$$[\delta_1,\delta_2]\begin{pmatrix} Z \\ P_L\chi \\ F \end{pmatrix} = -\frac{1}{2}\bar\epsilon_1\gamma^\mu\epsilon_2\partial_\mu\begin{pmatrix} Z \\ P_L\chi \\ F \end{pmatrix}$$

- Simplest action

$$S_{\rm kin} = \int {\rm d}^4x[-\partial^\mu\bar Z\partial_\mu Z - \bar\chi\partial\!\!\!/ P_L\chi + \bar F F]$$

- Potential term

$$S_F = \int {\rm d}^4x[FW'(Z) - \tfrac{1}{2}\bar\chi P_L W''(Z)\chi]$$

$$S = S_{\rm kin} + S_F + S_{\bar F}$$

# 6.2.2 The SUSY algebra

$$\delta(\epsilon)\phi(x) = \bar{\epsilon}\,\psi(x) = \epsilon^{\alpha}Q_{\alpha}\phi(x)$$
$$\rightarrow Q_{\alpha}\phi(x) = \psi_{\alpha}(x)$$

- A transformation is a parameter times a generator

$$\delta(\epsilon) = \epsilon^{\alpha}Q_{\alpha}$$

- Calculating a commutator

$$\begin{aligned}
\delta(\epsilon_1)\,\delta(\epsilon_2) &= \overbrace{\epsilon_1^{\alpha}Q_{\alpha}}^{\text{bosonic}}\epsilon_2^{\beta}Q_{\beta} \\
&= \epsilon_2^{\beta}\epsilon_1^{\alpha}Q_{\alpha}Q_{\beta}
\end{aligned}$$

$$[\delta(\epsilon_1),\,\delta(\epsilon_2)] = \epsilon_2^{\beta}\epsilon_1^{\alpha}Q_{\alpha}Q_{\beta} - \epsilon_1^{\alpha}\epsilon_2^{\beta}Q_{\beta}Q_{\alpha}$$
$$\begin{aligned}
&= \epsilon_2^{\beta}\epsilon_1^{\alpha}\left(Q_{\alpha}Q_{\beta} + Q_{\beta}Q_{\alpha}\right) \\
&= -\tfrac{1}{2}\bar{\epsilon}_1\gamma^{\mu}\epsilon_2\,\partial_{\mu}
\end{aligned}$$

# Calculating the algebra

- Very simple on *Z*
- On fermions: more difficult; needs Fierz rearrangement
- With auxiliary field: algebra satisfied for all field configurations
  Without auxiliary field: satisfied modulo field equations.
- auxiliary fields lead to
  - transformations independent of e.g. the superpotential
  - algebra universal : ‘closed off-shell’
  - useful in determining more general actions
  - in local SUSY: simplify couplings of ghosts

# 6.3. SUSY gauge theories

## 6.3.1 SUSY Yang-Mills vector multiplet

$$
\begin{aligned}
S_{\text{gauge}} &= \int \mathrm{d}^4x \left[-\tfrac{1}{4}F^{\mu\nu A}F^A_{\mu\nu} - \tfrac{1}{2}\bar{\lambda}^A\gamma^\mu D_\mu\lambda^A + \tfrac{1}{2}D^A D^A\right], \\
\delta A^A_\mu &= \tfrac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^A, \\
\delta\lambda^A &= \left[-\tfrac{1}{4}\gamma^{\rho\sigma}F^A_{\rho\sigma} + \tfrac{1}{2}\mathrm{i}\gamma_* D^A\right]\epsilon, \\
\delta D^A &= \tfrac{1}{2}\mathrm{i}\bar{\epsilon}\gamma_*\gamma^\mu D_\mu\lambda^A, \qquad D_\mu\lambda^A \equiv \partial_\mu\lambda^A + \lambda^C A_\mu{}^B f_{BC}{}^A
\end{aligned}
$$

$$
\begin{aligned}
\delta(\theta)A^A_\mu &= \partial_\mu\theta^A + \theta^C A_\mu{}^B f_{BC}{}^A, \\
\delta(\theta)\lambda^A &= \theta^C\lambda^B f_{BC}{}^A, \\
\delta(\theta)D^A &= \theta^C D^B f_{BC}{}^A
\end{aligned}
$$

$$
\begin{aligned}
[\delta_1,\delta_2]\,A^A_\mu &= -\tfrac{1}{2}\bar{\epsilon}_1\gamma^\nu\epsilon_2 F^A_{\nu\mu}, \\
[\delta_1,\delta_2]\,\lambda^A &= -\tfrac{1}{2}\bar{\epsilon}_1\gamma^\nu\epsilon_2 D_\nu\lambda^A, \\
[\delta_1,\delta_2]\,D^A &= -\tfrac{1}{2}\bar{\epsilon}_1\gamma^\nu\epsilon_2 D_\nu D^A .
\end{aligned}
$$

# 6.3.2 Chiral multiplets in SUSY gauge theories

Full theory

$$S = S_{\text{gauge}} + S_{\text{matter}} + S_{\text{coupling}} + S_W + S_{\overline{W}} \,.$$

$$\begin{aligned}
S_{\text{matter}} &= \int \mathrm{d}^4x \left[-D^\mu \bar{Z} D_\mu Z - \bar{\chi}\gamma^\mu P_L D_\mu \chi + \bar{F}F\right] , \\
S_{\text{coupling}} &= \int \mathrm{d}^4x \left[-\sqrt{2}(\bar{\lambda}^A \bar{Z} t_A P_L \chi - \bar{\chi} P_R t_A Z \lambda^A) + \mathrm{i}\, D^A \bar{Z} t_A Z\right] , \\
S_F &= \int \mathrm{d}^4x \left[F^\alpha W_\alpha + \tfrac{1}{2}\bar{\chi}^\alpha P_L W_{\alpha\beta}\chi^\beta\right] , \\
S_{\bar{F}} &= \int \mathrm{d}^4x \left[\bar{F}_\alpha \overline{W}^\alpha + \tfrac{1}{2}\bar{\chi}_\alpha P_R \overline{W}^{\alpha\beta}\chi_\beta\right] .
\end{aligned}$$

Modified chiral multiplet

$$\begin{aligned}
\delta Z &= \frac{1}{\sqrt{2}}\bar{\epsilon} P_L \chi \,, \\
\delta P_L \chi &= \frac{1}{\sqrt{2}} P_L(\gamma^\mu D_\mu Z + F)\epsilon \,, \\
\delta F &= \frac{1}{\sqrt{2}}\bar{\epsilon} P_R \gamma^\mu D_\mu \chi - \bar{\epsilon} P_R \lambda^A t_A Z \,,
\end{aligned}$$

# 6.4 Massless representations of $\mathcal{N}$-extended supersymmetry

Notation left-right

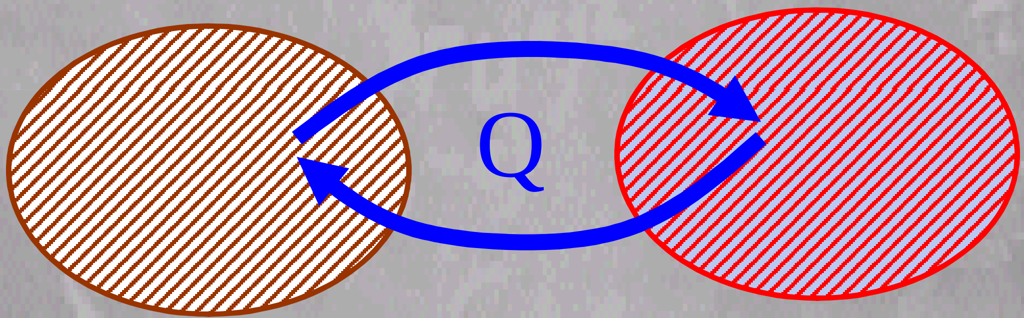
$$Q_i = P_L Q_i \,, \qquad Q^i = P_R Q^i \,.$$

Algebra:

$$\begin{aligned}
\left\{Q_{i\alpha}, \bar{Q}^{j\beta}\right\} &= -\tfrac{1}{2}\delta_i^j (P_L\gamma_\mu)_\alpha{}^\beta P^\mu \,, & \left\{Q^i_\alpha, \bar{Q}_j^\beta\right\} &= -\tfrac{1}{2}\delta_j^i (P_R\gamma_\mu)_\alpha{}^\beta P^\mu \,,\\
\left\{Q_{i\alpha}, \bar{Q}_j^\beta\right\} &= 0 \,, & \left\{Q^i_\alpha, \bar{Q}^{j\beta}\right\} &= 0 \,,\\
\left[M_{[\mu\nu]}, Q_{i\alpha}\right] &= -\tfrac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_{i\beta} \,, & \left[M_{[\mu\nu]}, Q^i_\alpha\right] &= -\tfrac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q^i_\beta \,,\\
\left[P_\mu, Q_{i\alpha}\right] &= 0 & \left[P_\mu, Q^i_\alpha\right] &= 0 \,.
\end{aligned}$$

# 6.4.1 Particle representations of $\mathcal{N}$ – extended supersymmetry

- There is an argument that
  \# bosonic d.o.f. = # fermionic d.o.f.,
  based on {Q,Q}=P (invertible)




- Should be valid for on-shell multiplets if eqs. of motion are satisfied: e.g. $z$ : 2, $\chi$ : 2 $\Rightarrow$ 2+2
- for off-shell multiplets counting all components: e.g. $z$ : 2, $\chi$ : 4, $h$ : 2 $\Rightarrow$ 4+4

# Spin content of representations of supersymmetry with maximal spin $s_{max} \leq 2$.

| | | s=2 | $s = 3/2$ | $s = 1$ | $s = 1/2$ | $s = 0$ |
|---|---|---|---|---|---|---|
| $\mathcal{N} = 1$ | $s_{max} = 2$ | 1 | 1 | | | |
| | $s_{max} = 3/2$ | | 1 | 1 | | |
| | $s_{max} = 1$ | | | 1 | 1 | |
| | $s_{max} = 1/2$ | | | | 1 | 1+1 |
| $\mathcal{N} = 2$ | $s_{max} = 2$ | 1 | 2 | 1 | | |
| | $s_{max} = 3/2$ | | 1 | 2 | 1 | |
| | $s_{max} = 1$ | | | 1 | 2 | 1 + 1 |
| | $s_{max} = 1/2$ | | | | 2 | 2 + 2 |
| $\mathcal{N} = 3$ | $s_{max} = 2$ | 1 | 3 | 3 | 1 | |
| | $s_{max} = 3/2$ | | 1 | 3 | 3 | 1+1 |
| | $s_{max} = 1$ | | | 1 | 3 +1 | 3 + 3 |
| $\mathcal{N} = 4$ | $s_{max} = 2$ | 1 | 4 | 6 | 4 | 1+1 |
| | $s_{max} = 3/2$ | | 1 | 4 | 6 +1 | 4 + 4 |
| | $s_{max} = 1$ | | | 1 | 4 | 6 |
| $\mathcal{N} = 5$ | $s_{max} = 2$ | 1 | 5 | 10 | 10 +1 | 5 + 5 |
| | $s_{max} = 3/2$ | | 1 | 5 +1 | 10 + 5 | 10 +10 |
| $\mathcal{N} = 6$ | $s_{max} = 2$ | 1 | 6 | 15 + 1 | 20 + 6 | 15 + 15 |
| | $s_{max} = 3/2$ | | 1 | 6 | 15 | 20 |
| $\mathcal{N} = 7$ | $s_{max} = 2$ | 1 | 7 + 1 | 21 + 7 | 35 + 21 | 35 + 35 |
| $\mathcal{N} = 8$ | $s_{max} = 2$ | 1 | 8 | 28 | 56 | 70 |

# Exercise on chapter 6

- Ex. 6.11 : Consider the theory of the chiral multiplet after elimination of $F$. Show that the action

$$S = \int \mathrm{d}^4x \left[ -\partial^\mu \bar{Z} \partial_\mu Z - \bar{\chi} \not{\partial} P_L \chi - \overline{W}' W' - \tfrac{1}{2} \bar{\chi} (P_L W'' + P_R \overline{W}'') \chi \right]$$

is invariant under the transformation rules

$$\begin{aligned}
\delta Z &= \tfrac{1}{\sqrt{2}} \bar{\epsilon} P_L \chi \,, & \delta \bar{Z} &= \tfrac{1}{\sqrt{2}} \bar{\epsilon} P_R \chi \\
\delta P_L \chi &= \tfrac{1}{\sqrt{2}} P_L (\not{\partial} Z + F) \epsilon, & \delta P_R \chi &= \tfrac{1}{\sqrt{2}} P_R (\not{\partial} \bar{Z} + \bar{F}) \epsilon \\
F &\equiv -\overline{W}'(\bar{Z}), & \bar{F} &= -W'(Z)
\end{aligned}$$

Show that the commutator on the scalar is still

$$[\delta_1, \delta_2] Z = -\tfrac{1}{2} \bar{\epsilon}_1 \gamma^\mu \epsilon_2 \partial_\mu Z$$

but is modified on the fermion as follows:

$$[\delta_1, \delta_2] P_L \chi = \bar{\epsilon}_1 \gamma^\mu \epsilon_2 P_L \left[ -\tfrac{1}{2} \partial_\mu \chi + \tfrac{1}{4} \gamma_\mu (\not{\partial} + \overline{W}'') \chi \right]$$

We find the spacetime translation plus an extra term that vanishes for any solution of the equations of motion.

# 7.9 Connections and covariant derivatives

$$\begin{aligned}\nabla_\mu V^\rho &= \partial_\mu V^\rho + \Gamma^\rho_{\mu\nu} V^\nu\,,\\ \nabla_\mu V_\nu &= \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu} V_\rho\,,\end{aligned}$$

metric postulate

$$\nabla_\mu g_{\nu\rho} \equiv \partial_\mu g_{\nu\rho} - \Gamma^\sigma_{\mu\nu} g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} g_{\nu\sigma} = 0$$

if there is no 'torsion' $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$

$$\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}(g) = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$$

# 7.12 Symmetries and Killing vectors

## 7.12.1 $\sigma$– model symmetries

Symmetries of action $S[\phi] = -\frac{1}{2}\int \mathrm{d}^D x\, g_{ij}(\phi)\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j$

can be parametrized as a general form $\delta(\theta)\phi^i = \theta^A k_A{}^i(\phi)$

Each $k_A{}^i$ (for every value of A) should satisfy

$$\nabla_i k_{jA} + \nabla_j k_{iA} = 0\,, \qquad k_{iA} = g_{ij}k_A^j\,, \qquad \nabla_i k_{jA} = \partial_i k_{jA} - \Gamma^k_{ij}(g)k_{kA}$$

Solutions are called 'Killing vectors' $k_A \equiv k_A^j \frac{\partial}{\partial\phi^j}$

and satisfy an algebra $[k_A\,,\,k_B] = f_{AB}{}^C k_C$

# 7.12.2 Symmetries of the Poincaré plane

Poincaré plane (X, Y>0)

$$\mathrm{d}s^2 = \frac{\mathrm{d}X^2 + \mathrm{d}Y^2}{Y^2} = \frac{\mathrm{d}Z\mathrm{d}\bar{Z}}{Y^2}$$

$\mathrm{SL}(2,\mathbb{R})$ transformations act as nonlinear maps

$$Z \to Z' = \frac{aZ+b}{cZ+d} = X' + \mathrm{i}Y'$$

# Exercise on chapter 7

- Ex. 7.48: Consider for the Poincaré plane $Z$ and $\bar{Z}$ as the independent fields, rather than $X$ and $Y$, and use the line element

$$\mathrm{d}s^2 = \frac{\mathrm{d}X^2 + \mathrm{d}Y^2}{Y^2} = \frac{\mathrm{d}Z\mathrm{d}\bar{Z}}{Y^2}$$

The metric components are

$$g_{ZZ} = g_{\bar{Z}\bar{Z}} = 0\,, \qquad g_{Z\bar{Z}} = g_{\bar{Z}Z} = -\frac{2}{(Z-\bar{Z})^2}$$

Show that the only non-vanishing components of the Christoffel connection are $\Gamma_{ZZ}{}^{Z}$ and its complex conjugate. Calculate them and then show that there are three Killing vectors,

$$k_1^Z = 1\,, \qquad k_2^Z = Z\,, \qquad k_3^Z = Z^2$$

each with conjugate. Show that their Lie brackets give a Lie algebra whose non-vanishing structure constants are

$$f_{12}{}^{1} = 1\,, \qquad f_{13}{}^{2} = 2\,, \qquad f_{23}{}^{3} = 1$$

This is a standard presentation of the Lie algebra of

$$\mathfrak{su}(1,1) = \mathfrak{so}(2,1) = \mathfrak{sl}(2)$$

# 12. Survey of supergravities

- To get an overview of what is possible
- and how geometry enters in supergravity

# 12.1 The minimal superalgebras
## 12.1.1. D=4

- Minimal algebra $\left\{Q_\alpha, Q_\beta\right\} = -\frac{1}{2}\gamma^a_{\alpha\beta}P_a$

$$[P, Q] = 0 \quad [M_{ab}, Q] = -\frac{1}{2}\gamma_{ab}Q$$

- Extension (using Weyl spinors and position of indices indicating chirality) $Q_i = P_L Q_i\,, \qquad Q^i = P_R Q^i\,.$

$$\left\{Q_{i\alpha}, Q^j_\beta\right\} = -\frac{1}{2}\delta^j_i (P_L\gamma^a)_{\alpha\beta}P_a\,,$$
$$\left\{Q_{i\alpha}, Q_{j\beta}\right\} = 0\,, \qquad \left\{Q^i_\alpha, Q^j_\beta\right\} = 0\,,$$

- Algebras exist for any $\mathcal{N}$.
  Field theory : $\mathcal{N} \leq 8$ i.e. at most 32 real supercharges.
  SUSY: $\mathcal{N} \leq 4$: 16 real supercharges

# 12.1.2. Minimal superalgebras in higher dimensions

$$\{Q_\alpha, Q_\beta\} = -\tfrac{1}{2}\gamma^a_{\alpha\beta}P_a$$

- is only consistent for $t_1 = -1$, i.e. Majorana
- previous can also be applied to D=8:
  but then only N=1 or N=2.
- Also same (without chirality)
  for D=9 (N=1 or N=2) and D=11 (N=1)
- D=10: supercharges can be chiral.
  The two Q's should have equal chirality
  - 1 chiral supercharge : "type I"
  - 2 of opposite chirality "type IIA"
  - 2 of same chirality: type IIB"

| D | | # |
|---|---|---|
| 4 | M | 4 |
| 5 | S | 8 |
| 6 | SW | 8 |
| 7 | S | 16 |
| 8 | M | 16 |
| 9 | M | 16 |
| 10 | MW | 16 |
| 11 | M | 32 |

# 12.2 The *R*-symmetry group

- Supersymmetries may rotate under an automorphism group. E.g. for 4 dimensions:

$$[T_A, Q_{\alpha i}\ ] = (U_A)_i{}^j Q_{\alpha j} \qquad [T_A, Q^i_\alpha\ ] = (U_A)^i{}_j Q^j_\alpha$$

- related by charge conjugation: $(U_A)^j{}_i = \left((U_A)_j{}^i\right)^*$

- Jacobi identities [*TTQ*] : *U* forms a representation of *T*-algebra
- Jacobi identities [*TQQ*] :

$$(U_A)_i{}^j = -(U_A)^j{}_i \equiv -\left((U_A)_j{}^i\right)^*.$$

$\rightarrow$ forms U($\mathcal{N}$) group

# *R*-symmetry groups

group that rotates susys: $\left[T_A, Q^i_\alpha\right] = (U_A)^i{}_j Q^j_\alpha$

- Majorana spinors in odd dimensions:
  SO(N) (D=3,9)
- Majorana spinors in even dimensions:
  U(N) (D=4,8)
- Majorana-Weyl spinors:
  $SO(N_L) \times SO(N_R)$ (D=2,10)
- Symplectic spinors:
  USp(N) (D=5,7)
- Symplectic Majorana-Weyl spinors:
  $USp(N_L) \times USp(N_R)$ (D=6)

# 12.4 Supergravity theories: towards a catalogue

- basic theories and kinetic terms

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\mathrm{bos}} &= \tfrac{1}{2}R + \tfrac{1}{4}(\mathrm{Im}\,\mathcal{N}_{AB})\mathcal{F}_{\mu\nu}^{A}\mathcal{F}^{\mu\nu B} \\
&\quad - \tfrac{1}{8}(\mathrm{Re}\,\mathcal{N}_{AB})e^{-1}\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}^{A}\mathcal{F}_{\rho\sigma}^{B} - \tfrac{1}{2}g_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j
\end{aligned}
$$

- deformations and gauged supersymmetry
  - covariant derivatives and field strengths
  - potential for the scalars

# The map: dimensions and # of supersymmetries

Strathdee,1987

| D | susy | 32 | 24 | 20 | 16 | 12 | 8 | 4 |
|---|---|---|---|---|---|---|---|---|
| 11 | M | M | | | | | | |
| 10 | MW | IIA, IIB | | | I | | | |
| 9 | M | N=2 | | | N=1 | | | |
| 8 | M | N=2 | | | N=1 | | | |
| 7 | S | N=4 | | | N=2 | | | |
| 6 | SW | (2,2) | (2,1) | | (1,1), (2,0) | | (1,0) | |
| 5 | S | N=8 | N=6 | | N=4 | | N=2 | |
| 4 | M | N=8 | N=6 | N=5 | N=4 | N=3 | N=2 | N=1 |
| | | SUGRA | | | SUGRA/SUSY | SUGRA | SUGRA/SUSY | |

vector multiplets

tensor multiplet

vector multiplets + multiplets up to spin 1/2

# 12.5 Scalars and geometry

- Scalar manifold can have isometries
  (symmetry of kinetic energy $ds^2 = g_{ij}\, d\phi^i\, d\phi^j$ )
- usually extended to symmetry of full action

  ('U-duality group')
- The connection between scalars and vectors in the matrix $\mathcal{N}_{AB}(\phi)$

  $\Rightarrow$ isometries act also as duality transformations
- A subgroup of the isometry group (at most of dimension $m$) can be gauged.

# Homogeneous / Symmetric manifolds

- If isometry group $G$ connect all points of a manifold $\rightarrow$ homogeneous manifold.
  Such a manifold can be identified with the coset $G/H$, where $H$ is the isotropy group: group of transformations that leave a point invariant
- If the algebras $\mathfrak{g}$ of G and $\mathfrak{h}$ of H have the structure

$$\forall g \in \mathfrak{g} \;:\; g = h + k\,, \qquad h \in \mathfrak{h}\,, \qquad k \in \mathfrak{k}\,,$$

$$\forall h_1, h_2 \in \mathfrak{h},\; k_1, k_2 \in \mathfrak{k} \;:\; [h_1,\, h_2] \in \mathfrak{h}\,, \qquad [h_1,\, k_1] \in \mathfrak{k}\,, \qquad [k_1,\, k_2] \in \mathfrak{h}$$

then the manifold is symmetric.

The curvature tensor is covariantly constant

# Geometries in supergravity

$$\mathcal{L} = \sqrt{g} g^{\mu\nu} (\partial_\mu \varphi^i)(\partial_\nu \varphi^j) g_{ij}(\varphi)$$

- Scalar manifolds for theories with more than 8 susys are symmetric spaces:
- Scalar manifolds for theories with 4 susys ($\mathcal{N}=1$, $D=4$, or lower $D$) are Kähler
- Scalar manifolds for theories with 8 susys are called 'special manifolds'.

  Include real, special Kähler, quaternionic manifolds

  They can be symmetric, homogeneous, or not even that

# The map of geometries

- With > 8 susys: symmetric spaces

| $d$ | 32 | 24 | 20 | 16 | | 12 |
|---|---|---|---|---|---|---|
| 9 | $\frac{S\ell(2)}{SO(2)} \otimes O(1,1)$ | | | $\frac{O(1,n)}{O(n)} \otimes O(1,1)$ | | |
| 8 | $\frac{S\ell(3)}{SU(2)} \otimes \frac{S\ell(2)}{U(1)}$ | | | $\frac{O(2,n)}{U(1)\times O(n)} \otimes O(1,1)$ | | |
| 7 | $\frac{S\ell(5)}{USp(4)}$ | | | $\frac{O(3,n)}{USp(2)\times O(n)} \otimes O(1,1)$ | | |
| 6 | $\frac{O(5,5)}{USp(4)\times USp(4)}$ | $\frac{SO(5,1)}{SO(5)}$ | | $\frac{O(4,n)}{O(n)\times SO(4)} \otimes O(1,1)$ | $\frac{O(5,n)}{O(n)\times USp(4)}$ | |
| 5 | $\frac{E_6}{USp(8)}$ | $\frac{SU^*(6)}{USp(6)}$ | | $\frac{O(5,n)}{USp(4)\times O(n)} \otimes O(1,1)$ | | |
| 4 | $\frac{E_7}{SU(8)}$ | $\frac{SO^*(12)}{U(6)}$ | $\frac{SU(1,5)}{U(5)}$ | $\frac{SU(1,1)}{U(1)} \times \frac{SO(6,n)}{SU(4)\times SO(n)}$ | | $\frac{SU(3,n)}{U(3)\times SU(n)}$ |

- 8 susys: very special, special Kähler and quaternionic spaces

| $d=6$ | $d=5$ | $d=4$ |
|---|---|---|
| $\frac{O(1,n)}{O(n)} \times \mathcal{QM}$ | $\mathcal{VSR} \times \mathcal{QM}$ | $\mathcal{SK} \times \mathcal{QM}$ |

**U(1)** part in holonomy group

**SU(2)=USp(2)** part in holonomy group

- 4 susys: Kähler: **U(1)** part in holonomy group

# Exercise on Chapter 12

- Ex.12.3 Consider an arbitrary point in the Poincaré plane and find the Killing vector $c^A k_A$ that vanishes. Check that the other two Killing vectors in that point are independent.
- Ex.12.4 Why do the isotropy generators define a group? How do you associate the manifold to the coset space?
- Ex 12.5 Check that the Poincaré plane is a symmetric space.

# 13. Complex manifolds

## 13.1 The local description of complex and Kähler manifolds

- Use complex coordinates

$$\{z^a\} = \{z^\alpha, \bar{z}^{\bar{\alpha}}\} \qquad a = 1, \ldots, 2n;\ \alpha, \bar{\alpha} = 1, \ldots, n$$

$$\mathrm{d}s^2 = 2 g_{\alpha\bar{\beta}} \mathrm{d}z^\alpha \mathrm{d}\bar{z}^{\bar{\beta}} \quad ..$$

Hermitian manifold

define fundamental 2-form $\quad \Omega = -2\mathrm{i} g_{\alpha\bar{\beta}} \mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^{\bar{\beta}}$

Kähler manifold: closed fundamental 2-form

$$\mathrm{d}\Omega = -\mathrm{i}(\partial_\gamma g_{\alpha\bar{\beta}} - \partial_\alpha g_{\gamma\bar{\beta}}) \mathrm{d}z^\gamma \wedge \mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^{\bar{\beta}} + \mathrm{c.c.} = 0$$

# Properties of metric, connection, curvature for Kähler manifolds

- metric derivable from a 'Kähler potential'

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K(z, \bar{z})$$

- connections have only unmixed components

$$\Gamma^\alpha_{\beta\gamma} = g^{\alpha\bar{\delta}} \partial_\beta g_{\gamma\bar{\delta}} \, , \qquad \Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}} = g^{\delta\bar{\alpha}} \partial_{\bar{\beta}} g_{\delta\bar{\gamma}} .$$

- curvature components related to

$R_{\bar{\delta}\gamma}{}^{\alpha}{}_{\beta} = \partial_{\bar{\delta}} \Gamma^\alpha_{\beta\gamma}$ (two holomorphic indiees up and down, and symmetric in these pairs)

- Ricci tensor $R_{ab} = g^{cd} R_{acbd} = R_{ba}$

$$R_{\alpha\bar{\beta}} = g^{\bar{\gamma}\gamma} R_{\alpha\bar{\gamma}\bar{\beta}\gamma} = -R_{\alpha\bar{\beta}\gamma}{}^{\gamma} = -\partial_\alpha \partial_{\bar{\beta}} (\log \det g_{\gamma\bar{\delta}})$$

# 13.2 Mathematical structure of Kähler manifolds

- starts from a complex structure
  - almost complex: tensor on tangent space $J_i{}^k\, J_k{}^j = -\,\delta_i^j$
  - Nijenhuis tensor vanishes. In presence of a torsion-free connection, this is implied by covariant constancy of complex structure $$\nabla_k J_i{}^j = \partial_k J_i{}^j - \Gamma_{ki}^{\ell} J_\ell{}^j + \Gamma_{k\ell}^{j} J_i{}^\ell = 0$$
  - metric hermitian : $JgJ^T=g$ and Levi-Civita connection of this metric is used above
- Then the Kähler form is $\Omega = -J_{ij}\mathrm{d}\phi^i \wedge \mathrm{d}\phi^j\,, \qquad J_{ij} = J_i{}^k g_{kj}$
- In complex coordinates

$$J = \begin{pmatrix} \mathrm{i}\delta_\alpha{}^\beta & 0 \\ 0 & -\mathrm{i}\delta_{\bar\alpha}{}^{\bar\beta} \end{pmatrix}.$$

# 13.4 Symmetries of Kähler metrics

## 13.4.1 Holomorphic Killing vectors and moment maps

$$\delta\phi^i = \theta k^i(\phi) \qquad \text{or} \qquad \delta z^\alpha = \theta k^\alpha(z,\bar z)$$

- require vanishing Lie derivatives of metric *and* of complex structure.
- Implies that in complex coordinates
  - the Killing vector is holomorphic
  - Lie derivative of Killing form vanishes
    
    $\rightarrow$ Killing vectors determined by real moment map $\mathcal{P}$

PS: a Kähler manifold is a symplectic manifold due to the existence of the Kähler 2-form. Moment map is generating function of a canonical transformation

$$\begin{aligned}
0 &= \mathcal{L}_k\Omega = (i_k\mathrm{d} + \mathrm{d}i_k)\,\Omega = \mathrm{d}i_k\Omega \\
i_k\Omega &= -2\mathrm{d}\mathcal{P} \\
k_\alpha &= g_{\alpha\bar\beta}\,k^{\bar\beta}(\bar z) = \mathrm{i}\partial_\alpha\,\mathcal{P}(z,\bar z)\,, \\
k_{\bar\alpha} &= g_{\beta\bar\alpha}\,k^{\beta}(z) = -\mathrm{i}\partial_{\bar\alpha}\,\mathcal{P}(z,\bar z)\,.
\end{aligned}$$

# Kähler transformations and the moment map

- Kähler potential is not unique: $g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K(z,\bar{z})$
- Kähler transformations

$$K(z,\bar{z}) \to K'(z,\bar{z}) = K(z,\bar{z}) + f(z) + \bar{f}(\bar{z})$$

- Also for symmetries

$$\delta K = \theta \left( k^\alpha \partial_\alpha + k^{\bar{\alpha}} \partial_{\bar{\alpha}} \right) K(z,\bar{z}) = \theta \left[ r(z) + \bar{r}(\bar{z}) \right]$$

$$\mathcal{P}(z,\bar{z}) = \mathrm{i} \left[ k^\alpha \partial_\alpha K(z,\bar{z}) - r(z) \right] = -\mathrm{i} \left[ k^{\bar{\alpha}} \partial_{\bar{\alpha}} K(z,\bar{z}) - \bar{r}(\bar{z}) \right] .$$

# Exercises on chapter 13

- Ex. 13.14: Show that the metric of the Poincaré plane of complex dimension 1 is a Kähler metric.
  What is the Kähler potential?
- Ex. 13.18: Consider CP$^1$ with Killing potential $K = \ln(1 + z\bar{z})$
  - Check that there are 3 Killing vectors

$$k_1 = -\mathrm{i}\frac{1}{2}(1 - z^2)\frac{\partial}{\partial z}\,,$$

$$k_2 = \frac{1}{2}(1 + z^2)\frac{\partial}{\partial z}\,,$$

$$k_3 = -\mathrm{i}z\frac{\partial}{\partial z}\,.$$

  - that satisfy the su(2) algebra

$$[k_A\,,\,k_B] = \varepsilon_{ABC}k_C$$

- Ex. 13.20: Apply $\delta K = \theta^A \left(k_A^z \partial_z + k_A^{\bar{z}}\partial_{\bar{z}}\right) K(z,\bar{z}) = \theta^A \left[r_A(z) + \bar{r}_A(\bar{z})\right]$ to obtain

$$r_1 = \tfrac{1}{2}\mathrm{i}z\,, \qquad r_2 = \tfrac{1}{2}z\,, \qquad r_3 = -\tfrac{1}{2}\mathrm{i}$$

Note that the Kähler potential is invariant under $k_3$, but still $r_3 \neq 0$.
Its value is fixed by the 'equivariance relation'

$$k_A{}^\alpha g_{\alpha\bar{\beta}} k_B{}^{\bar{\beta}} - k_B{}^\alpha g_{\alpha\bar{\beta}} k_A{}^{\bar{\beta}} = \mathrm{i} f_{AB}{}^C \mathcal{P}_C$$

# 14. General actions with $\mathcal{N}$=1 supersymmetry

## 14.1 Multiplets

- Multiplets are sets of fields on which the supersymmetry algebra is realized.
- A chiral multiplet is a multiplet in which the transformation of the lowest (complex scalar) component involves only $P_L\epsilon$.
- A real multiplet is a multiplet in which the lowest component is a real scalar.
- Allowing general SUSY transformations with these requirements determines the multiplet

## 14.1.1 Chiral multiplets

$$\delta Z = \frac{1}{\sqrt{2}}\bar{\epsilon}P_L\chi,$$
$$\delta P_L\chi = \frac{1}{\sqrt{2}}P_L(\partial\!\!\!/ Z + F)\epsilon,$$
$$\delta F = \frac{1}{\sqrt{2}}\bar{\epsilon}\,\partial\!\!\!/ P_L\chi$$

## 14.1.2 Real multiplets

$$\delta C = \tfrac{1}{2}\mathrm{i}\bar{\epsilon}\gamma_*\zeta,$$
$$\delta P_L\zeta = \tfrac{1}{2}P_L\left(\mathrm{i}\mathcal{H} - B\!\!\!/ - \mathrm{i}\partial\!\!\!/ C\right)\epsilon,$$
$$\delta\mathcal{H} = -\mathrm{i}\bar{\epsilon}P_R\left(\lambda + \partial\!\!\!/\zeta\right),$$
$$\delta B_\mu = -\tfrac{1}{2}\bar{\epsilon}\left(\gamma_\mu\lambda + \partial_\mu\zeta\right),$$
$$\delta\lambda = \tfrac{1}{2}\left[\gamma^{\rho\sigma}\partial_\rho B_\sigma + \mathrm{i}\gamma_* D\right]\epsilon,$$
$$\delta D = \tfrac{1}{2}\mathrm{i}\,\bar{\epsilon}\gamma_*\gamma^\mu\partial_\mu\lambda.$$

gauge multiplet is a submultiplet:
a real multiplet with only components invariant under
a supergauge transformation $C \to C + \mathrm{Im}\, Z$
Wess-Zumino gauge : $C=\zeta=\mathcal{H}=0$

# 14.2 Generalized actions by multiplet calculus

$$S_F = \int \mathrm{d}^4x\, F, \qquad S_D = \int \mathrm{d}^4x\, D$$

are invariant under SUSY,

$$\delta S_F = \int \mathrm{d}^4x\, \delta F = \frac{1}{\sqrt{2}} \int \mathrm{d}^4x\, \bar\epsilon \partial\!\!\!/ P_L \chi = 0,$$

$$\delta S_D = \int \mathrm{d}^4x\, \delta D = \frac{1}{2}\mathrm{i} \int \mathrm{d}^4x\, \bar\epsilon \gamma_* \partial\!\!\!/ \lambda = 0.$$

Can be applied to ‘composite multiplets’ constructed from elementary ones.

Reality : $D$ is real, but $F$ is complex: add complex conjugate

Terminology:

*F*-type actions for composite chiral multiplets,

*D*-type actions for composite real multiplets

## 14.2.1 The superpotential

- Start with *W(Z)*: is chiral $\longrightarrow$ *F*-action

## 14.2.2 Kinetic terms for chiral multiplets

- Start from $K(Z,\bar{Z})$ : is real $\longrightarrow$ *D*-action

## 14.2.3 Kinetic terms for gauge multiplets

- See that $P_L\,\lambda$ transforms chirally
- Start with $f_{AB}(Z)\bar{\lambda}^A P_L \lambda^B$: is chiral $\longrightarrow$ *F*-action

$$\boxed{\mathcal{L} = [K(Z,\bar{Z})]_D + [W(Z)]_F + \left[f_{AB}(Z)\bar{\lambda}^A P_L \lambda^B\right]_F}$$

# 14.3 Kähler geometry from chiral multiplets

Kähler metric

$$
\begin{aligned}
D(\tfrac{1}{2}K) \;=\; & K_{\alpha\bar\beta}\left(-\partial_\mu Z^\alpha \partial^\mu \bar Z^{\bar\beta} - \tfrac{1}{2}\bar\chi^\alpha P_L \not{\partial}\chi^{\bar\beta} - \tfrac{1}{2}\bar\chi^{\bar\beta} P_R \not{\partial}\chi^\alpha + F^\alpha \bar F^{\bar\beta}\right) \\
& + \tfrac{1}{2}\left[K_{\alpha\beta\bar\gamma}\left(-\bar\chi^\alpha P_L \chi^\beta \bar F^{\bar\gamma} + \bar\chi^\alpha P_L(\not{\partial} Z^\beta)\chi^{\bar\gamma}\right) + \text{h.c.}\right] \\
& + \tfrac{1}{4}K_{\alpha\beta\bar\gamma\bar\delta}\,\bar\chi^\alpha P_L \chi^\beta \bar\chi^{\bar\gamma} P_R \chi^{\bar\delta}.
\end{aligned}
$$

elimination of auxiliary fields

$$
F^\alpha = \tfrac{1}{2} g^{\alpha\bar\beta} K_{\gamma\beta\bar\beta}\bar\chi^\gamma P_L \chi^\beta = \tfrac{1}{2}\Gamma^\alpha_{\gamma\beta}\bar\chi^\gamma P_L \chi^\beta .
$$

$$
\begin{aligned}
S(K)|_F \;=\; & \int \mathrm{d}^4 x \left[g_{\alpha\bar\beta}\left(-\partial_\mu Z^\alpha \partial^\mu \bar Z^{\bar\beta} - \tfrac{1}{2}\bar\chi^\alpha P_L \not{\nabla}\chi^{\bar\beta} - \tfrac{1}{2}\bar\chi^{\bar\beta} P_R \not{\nabla}\chi^\alpha\right)\right. \\
& \left. + \tfrac{1}{4}R_{\alpha\bar\gamma\beta\bar\delta}\,\bar\chi^\alpha P_L \chi^\beta \bar\chi^{\bar\gamma} P_R \chi^{\bar\delta}\right].
\end{aligned}
$$

$$
P_L \nabla_\mu \chi^\alpha \equiv P_L\left(\partial_\mu \chi^\alpha + \Gamma^\alpha_{\beta\gamma}\chi^\gamma \partial_\mu Z^\beta\right)
$$